Kuwait University | MATH 102 | May, 31, 2005

Math. Dept. | **Final Examination** | Duration 2 hours

Calculators and mobile telephones are not allowed.

Answer the following questions.

1. (3 points) Let $f(x) = \cos^{-1}(e^x) - \ln(x+2) - \cos^{-1}(e^{-1})$.

(a) Find the domain of f.

(b) Show that f has an inverse.

(c) Find the slope of the tangent line to the graph of f^{-1} at the point $P(0, -1)$.

2. (2+2 points)

(a) Find $\lim_{x\to 0} \left(\frac{\sinh 2x}{\sin x}\right)^{\tanh x}$

(b) Determine all x for which $\log_{\frac{1}{2}}(x^2 - 3) < 0$.

3. Evaluate the following integrals: **(4 points each)**

(a) $\int \frac{x+1}{x^2+5x+6}dx$ (b) $\int \frac{x+\sin x}{\cos^2 x}dx$ (c) $\int (1+\cos 2x)^{\frac{3}{2}}dx$

4. (3 points) Evaluate if possible $\int_0^{\infty} \frac{xdx}{x^4+1}$.

5. (4 points) Find the surface area of the solid generated by rotating the curve $y = \cosh x$, $0 < x < 3$ about the x-axis.

6. (4 points) Find the centroid of the region bounded by the curves $y = 1 - x^2$, $y = 0$.

7. (4+2 points) Let C be the curve given by parametric equations: $x(t) = \cosh\sqrt{t}$, $y(t) = 1 + \sqrt{t}$, $1 \le t \le 9$.

(a) Find the length of C.

(b) Find the slope of the tangent line to C at the point corresponding to $t = 4$.

8. (4 points) Find the area inside the graphs of both of the polar equations $r = 3\cos\theta$, $r = 1 + \cos\theta$.

SOLUTIONS

May 31, 2005

1. (a) $-1 \le e^x \le 1 \iff x \in (-\infty, 0]$; $x+2>0 \iff x \in (-2,\infty)$ $\Rightarrow$ Domain $(-2,($

(b) $f'(x) = -\dfrac{e^x}{\sqrt{1-e^{2x}}} - \dfrac{1}{x+2} < 0 \Rightarrow f$ is decreasing $\Rightarrow f$ is 1-1 $\Rightarrow \exists f^{-1}$

(c) $m = (f^{-1})'(0) = \dfrac{1}{f'(-1)} = -\dfrac{\sqrt{e^2-1}}{1+\sqrt{e^2-1}}$.

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2. (a) $\lim\limits_{x\to 0} \dfrac{\sinh 2x}{\sin x} = \dfrac{0}{0}$; Apply L'H Rule : $\lim\limits_{x\to 0}\dfrac{\sinh 2x}{\sin x} = \lim\limits_{x\to 0}\dfrac{2\cosh 2x}{\cos x}$
$= 2$. As $\lim\limits_{x\to 0} \tanh x = 0$, we have : $\lim\limits_{x\to 0}\left(\dfrac{\sinh 2x}{\sin x}\right)^{\tanh x} = 2^0 = 1$.

(b) $\log_{\frac{1}{2}}(x^2-3) < 0 \iff \log_{\frac{1}{2}}(x^2-3) < \log_{\frac{1}{2}} 1 \iff x^2-3 > 1 \iff x^2-4>0$.
So : $x \in (-\infty,-2) \cup (2,\infty)$.

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3. (a) $\dfrac{x+1}{x^2+5x+6} = \dfrac{2}{x+3} - \dfrac{1}{x+2} \Rightarrow \displaystyle\int \frac{x+1}{x^2+5x+6}dx = \ln\frac{(x+3)^2}{|x+2|} + C$.

(b) $\displaystyle\int \frac{x}{\cos^2 x}dx = \int x\sec^2 x\, dx = x\tan x - \int \tan x\, dx = x\tan x + \ln|\cos$
$(u=x\ ,\ dv = \sec^2 x\, dx\ ,\ du = dx,\ v=\tan x)$

$\displaystyle\int \frac{\sin x}{\cos^2 x}dx = -\int\frac{dt}{t^2} = \frac{1}{t} = \sec x$. So: $\displaystyle\int\frac{x+\sin x}{\cos^2 x}dx = x\tan x + \ln|\cos x| + \sec x + ($
$(t = \cos x,\ \ dt = -\sin x\, dx)$

(c) $\displaystyle\int (1+\cos 2x)^{3/2}dx = \int (2\cos^2 x)^{3/2}dx = 2\sqrt{2}\int\cos^3 x\, dx = 2\sqrt{2}\left(\sin x - \frac{\sin^3 x}{3}\right) +$

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4. $\displaystyle\int\frac{x\,dx}{x^4+1} = \frac{1}{2}\int\frac{du}{1+u^2} = \frac{1}{2}\tan^{-1}(x^2)$; $\displaystyle\lim_{t\to\infty}\int_0^t\frac{x\,dx}{x^4+1} = \frac{1}{2}\lim_{t\to\infty}\tan^{-1}(x^2)\Big|_0^t$
$(x^2 = u\ ,\ 2x\,dx = du)$
$= \dfrac{1}{2}\lim\limits_{t\to\infty}\left(\tan^{-1}(t^2) - \tan^{-1}(0)\right) = \dfrac{1}{2}\left(\dfrac{\pi}{2}-0\right) = \dfrac{\pi}{4}$. $\Rightarrow \displaystyle\int_0^\infty \frac{x\,dx}{x^4+1} = \frac{\pi}{4}$.

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5. Area $(R) = \displaystyle\int_0^3 2\pi y\sqrt{1+(y')^2}\,dx = 2\pi\int_0^3 \cosh x\sqrt{1+\sinh^2 x}\,dx = 2\pi\int_0^3\cosh^2 x\, d$
$= \pi\displaystyle\int_0^3 (1+\cosh 2x)dx = \pi\left(x + \frac{1}{2}\sinh 2x\right)\Big|_0^3 = 3\pi + \frac{\pi}{2}\sinh 6$.

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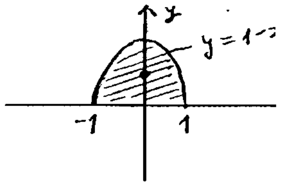
6. $A = \displaystyle\int_{-1}^{1}(1-x^2)dx = \frac{4}{3}$. $\bar{x} = \dfrac{1}{A}\displaystyle\int_{-1}^{1}x(1-x^2)dx = 0$,



$\bar{y} = \dfrac{1}{2A}\displaystyle\int_{-1}^{1}(1-x^2)^2dx = \frac{3}{2}\int_0^1(1-2x^2+x^4)dx = \frac{2}{5}$.
$\Rightarrow G\left(0, \frac{4}{5}\right)$.

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7. (a) $\dfrac{dx}{dt} = \dfrac{1}{2\sqrt{t}}\sinh\sqrt{t}$, $\dfrac{dy}{dt} = \dfrac{1}{2\sqrt{t}}$ $\Rightarrow L = \displaystyle\int_1^9\sqrt{\frac{1}{4t}(\sinh^2\sqrt{t}+1)}\,dt$
$= \dfrac{1}{2}\displaystyle\int_1^9\frac{\cosh\sqrt{t}}{\sqrt{t}}dt = \sinh\sqrt{t}\,\Big|_1^9 = \sinh 3 - \sinh 1$.

(b) $m = \dfrac{dy/dt}{dx/dt}\Big|_{t=4} = \dfrac{1}{\sinh\sqrt{t}}\Big|_{t=4} = \dfrac{1}{\sinh 2}$.

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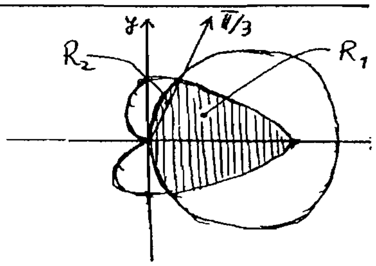
8. Area $(R) = 2\,($Area$(R_1) +$ Area$(R_2))$



$\displaystyle\int_0^{\pi/3}(1+\cos\theta)^2d\theta + \int_{\pi/3}^{\pi/2}9\cos^2\theta\,d\theta$
$= \displaystyle\int_0^{\pi/3}(1+2\cos\theta+\cos^2\theta)d\theta + 9\int_{\pi/3}^{\pi/2}\cos^2\theta\,d\theta$
$= \displaystyle\int_0^{\pi/3}\left(\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta\right)d\theta + \frac{9}{2}\int_{\pi/3}^{\pi/2}(1+\cos 2\theta)d\theta$
$= \left(\dfrac{3}{2}\theta + 2\sin\theta + \dfrac{1}{4}\sin 2\theta\right)\Big|_0^{\pi/3} + \dfrac{9}{2}\left(\theta + \dfrac{1}{2}\sin 2\theta\right)\Big|_{\frac{\pi}{3}}^{\pi/2} = \dfrac{5\pi}{4}$.